— Area Formula —

§1. Hausdorff Measure

For $A \subseteq \mathbb{R}^n$, $0 \le s < \infty$, $0 < \delta$, define

$$\mathcal{H}^s_\delta(A) = \inf\left\{\alpha \sum \left(\frac{d(C_j)}{2}\right)^s : A \subset \bigcup C_j,\ d(C_j) < \delta\right\}$$

where $\alpha = \alpha(s)$ is a normalizing constant. ($\alpha(n) r^n$ = volume of the unit ball in $\mathbb{R}^n$)

$$\mathcal{H}^s(A) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(A)$$
$$= \sup_{\delta > 0} \mathcal{H}^s_\delta(A).$$

Properties:

~ Borel regular measure, $\forall A \subset \mathbb{R}^n$, $\exists$ Borel $\tilde{A}$, $A \subset \tilde{A}$ s.t. $\mathcal{H}^s(A) = \mathcal{H}^s(\tilde{A})$

~ $\mathcal{H}^0$ = counting measure, $\mathcal{H}^n = \mathcal{L}^n$ in $\mathbb{R}^n$.

~ $\mathcal{H}^s(\lambda A) = \lambda^s \mathcal{H}^s(A)$, $\lambda > 0$

~ $\mathcal{H}^s(LA) = \mathcal{H}^s(A)$, L Euclidean motion, $Lx = Ax + b$ (orthogonal matrix; translat-)

~ $\mathcal{H}^s$ non-Radon in $\mathbb{R}^n$ for $0 \le s < n$.

§2. Under Lipschitz contin. Maps.

$f : \mathbb{R}^n \to \mathbb{R}^m$ is Lipschitz continuous. Define

Tel: 2788 8524 Fax: 2788 9315 Web Site: http://www.cityu/edu.hk/col

$$\operatorname{Lip} f = \sup_{\substack{x,y \\ x \neq y}} \frac{|f(x)-f(y)|}{|x-y|}$$

(A more common notation is $[f]_{C^{0,1}}$.) Bdd Lip. conti fcns on $\mathbb{R}^n$ form $C^{0,1}(\mathbb{R}^n)$, a Banach space under the norm

$$\begin{aligned} \|f\|_{C^{0,1}} &= \sup|f(x)| + \operatorname{Lip} f \\ &= \|f\|_{L^\infty} + [f]_{C^{0,1}}. \end{aligned}$$

It can be identified with the Sobolev space $W^{1,\infty}(\mathbb{R}^n)$.

Proposition 1 Let f be Lip. conti $\mathbb{R}^n \to \mathbb{R}^m$. Then for $A \subset \mathbb{R}^n$

$$\mathcal{H}^s(f(A)) \leq (\operatorname{Lip} f)^s \mathcal{H}^s(A), \quad 0 \leq s \leq n.$$

For later use, we consider linear maps in certain dimensions.

Proposition 2

(a) $T: \mathbb{R}^n \to \mathbb{R}^n$ linear, $\mathcal{L}^n(TA) = |\det T| \mathcal{L}^n(A)$ where $\det T$ is the determinant of any matrix rep. of T.

(b) $T: \mathbb{R}^n \to \mathbb{R}^m$, $n \leq m$, orthogonal

$$\mathcal{H}^n(TA) = \mathcal{L}^n(A).$$

(c) $T: \mathbb{R}^n \to \mathbb{R}^m$, $n \leq m$, linear

$$\mathcal{H}^n(TA) = \|T\| \mathcal{L}^n(A), \quad \text{where}$$

$\|T\| = |\det S|$, $T = O \circ S$ in polar deomposition.

Pf: (a) See [R1] (2.19 – 2.23)

(b) Given $\varepsilon > 0$ and $\delta > 0$, find $\{C_j\}_\Lambda$ s.t. $TA \subset \bigcup C_j$, $d(C_j) < \delta$,

$$\mathcal{H}^n_\delta(TA) + \varepsilon \geq \alpha(n) \sum \left(\frac{d(C_j)}{2}\right)^n$$

$$\geq \alpha(n) \sum \left(\frac{d(C_j \cap TA)}{2}\right)^n$$

$$= \alpha(n) \sum \left(\frac{d(\tilde{C}_j)}{2}\right)^n, \quad \tilde{C}_j = T^{-1}(C_j \cap TA),$$

$$\geq \mathcal{H}^n_\delta(A).$$ Now let $\varepsilon \to 0$ and then $\delta \to 0$.

Note the orthogonal map preserves distance and diameter.

(c) Combine (a) and (b).

§3. Area Formula

<u>Proposition 3</u>. $A \subseteq \mathbb{R}^n$ <u>$\mathcal{L}^n$-measurable</u>. <u>Then</u>

(a) $f(A)$ <u>$\mathcal{H}^n$-measurable</u>.

(b) $y \mapsto \mathcal{H}^0(A \cap f^{-1}\{y\})$ $\mathcal{H}^n$-measurable in $\mathbb{R}^m$,

(c)
$$\int_{\mathbb{R}^m} \mathcal{H}^0(A \cap f^{-1}\{y\})\, d\mathcal{H}^n \leq (\mathrm{Lip} f)^n \mathcal{L}^n(A)$$

Pf: See [EG] 3.3.

Proposition 4 For $t>1$,

$$B=\{x: Df(x) \text{ exists}, \ Jf(x)>0\}.$$

$\exists \{E_k\}$ Borel sets in $\mathbb{R}^n$ s.t.

(i) $B=\bigcup_1^\infty E_k$,

(ii) $f|_{E_k}$ is 1-1

(iii) $\exists T_k : \mathbb{R}^n \to \mathbb{R}^n$ symmetric nonsingular map

$$\text{Lip}(f|_{E_k} \circ T_k^{-1}), \ \text{Lip}(T_k \circ (f|_{E_k})^{-1}) \le t \qquad (1)$$

$$t^{-n}|\det T_k| \le Jf|_{E_k} \le t^n |\det T_k|, \ \forall k. \qquad (2)$$

Pf. Fix $\varepsilon>0$ small s.t. $\frac{1}{t}+\varepsilon<1<t-\varepsilon$.

Let $\mathcal{C}$ be a countable dense set in B and $\mathcal{S}$ a countable dense set in all symmetric automorphisms of $\mathbb{R}^n$.

$\forall c \in \mathcal{C}, T \in \mathcal{S}, i \in \mathbb{N}$, set

$$E(c,T,i)=\{b : b \in B \cap B_{1/i}(c),$$

$$\left(\tfrac{1}{t}+\varepsilon\right)|Tv| \le |Df(b)v| \le (t-\varepsilon)|Tv|, \ \forall v \qquad (3)$$

$$|f(a)-f(b)-Df(b)\cdot(a-b)| \le \varepsilon|T(a-b)|, \ \forall a \in B_{2/i}(b). \qquad (4)$$

Ex: Show $E(c,T,i) \in \mathcal{B}$.

Roughly speaking, $E(c,T,i)$ consists of those b which are close to c in the order of $1/i$ and whose differentials are close to T as described in ③ and ④.

Note that ③ + ④ $\Rightarrow$ $\forall a \in B_{2/i}(b)$, $b \in E(c,T,i)$

$$\frac{1}{t}|T(a-b)| \le |f(a)-f(b)| \le t|T(a-b)| \quad ⑤$$

Claim: $\forall b \in E(c,T,i)$,

$$\left(\frac{1}{t}+\varepsilon\right)^n |\det T| \le Jf(b) \le (t-\varepsilon)^n |\det T| \quad ⑥$$

Pf: $Df(b) = O \circ S$ (polar decomposition), so

$$Jf(b) = |\det S|.$$

By ③, $\left(\frac{1}{t}+\varepsilon\right)|Tv| \le |Sv| \le (t-\varepsilon)|Tv| \Rightarrow$

$$\left(\frac{1}{t}+\varepsilon\right)|v| \le |S \circ T^{-1} v| \le (t-\varepsilon)|v|, \quad \forall v$$

$\Rightarrow$ $B_{\frac{1}{t}+\varepsilon}(0) \subset (S \circ T^{-1})(B_1(0)) \subset B_{t-\varepsilon}(0)$, so ⑥ follows from $|\det(S \circ T^{-1})| = Jf(b)/|\det T|$.

We order $E(c,T,i)$ into $\{E_k\}$.

Pf of (i): $B \subset \bigcup E_k$. For $b \in B$, let $Df(b) = O \circ S$.

First choose $T \in \mathcal{D}$ s.t. $\mathrm{Lip}(T \circ S^{-1}) \le \dfrac{1}{\frac{1}{t}+\varepsilon}$,

$$\mathrm{Lip}(S \circ T^{-1}) \le t - \varepsilon.$$

Possible $\because$ $\mathcal{D}$ is dense.

Next, some large i, s.t.

$$|f(a)-f(b)-Df(b)\cdot(a-b)| \leq \frac{\varepsilon}{Lip(T^{-1})}|a-b|,$$

$$\forall a \in B_{2/i}(b)$$

($\because$ f is diff. at b)

Finally, find c s.t. $b \in B_{1/i}(c)$. Then (3) follows from our choice of T, and (4) follows from above since

$$\frac{\varepsilon}{Lip(T^{-1})}|a-b| = \frac{\varepsilon}{Lip(T^{-1})}|T^{-1}T(a-b)|$$

$$\leq \frac{\varepsilon\, Lip(T^{-1})}{Lip(T^{-1})}|T(a-b)|$$

$$= \varepsilon|T(a-b)|.$$

We've shown (i) holds.

Next, observe that

$$E(c,T,i) \subset B_{2/i}(b) \quad \text{for } b \in E(c,T,i)$$

($b' \in E(c,T,i) \Rightarrow b' \in B_{1/i}(c)$. As $c \in B_{1/i}(b)$, $b' \in B_{2/i}(b)$, done.)

So, we can take $a = b'$ in (5) to conclude $f|_{E(c,T,i)}$ is 1-1.

(ii) holds.

Finally, (iii) follows from the choice of T and (6) (using $E(c,T,i) \subset B_{2/i}(b)$.) #

Theorem 5 $f: \mathbb{R}^n \to \mathbb{R}^m$, $n \le m$, Lip. continuous.

$\forall$ $\mathcal{L}^n$-measurable $A \subset \mathbb{R}^n$,

$$\int_A Jf \, d\mathcal{L}^n = \int_{\mathbb{R}^m} \mathcal{H}^0(A \cap f^{-1}\{y\}) \, d\mathcal{H}^n(y).$$

The proof is written in details in [EG].